

# Shift versus Extension in Refined Partition Functions

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We have recently shown that the global behavior of the partition function of  $\mathcal{N} = 2$  gauge theory in the general  $\Omega$ -background is captured by special geometry in the guise of the (extended) holomorphic anomaly equation. We here analyze the fate of our results under the shift of the mass parameters of the gauge theory. The preferred value of the shift, noted previously in other contexts, restores the  $\mathbb{Z}_2$  symmetry of the instanton partition function under inversion of the  $\Omega$ -background, and removes the extension. We comment on various connections.

## I. INTRODUCTION

In recent work [1], we have initiated the systematic study of refined (topological string, or gauge theory in the general  $\Omega$ -background) partition functions from the point of view of the special geometry governing the underlying moduli space.

Starting from the explicit expressions for the gauge theory partition function  $Z(a, m, \epsilon_1, \epsilon_2; q)$  which was obtained in [2] using localization on the moduli space of instantons, and parametrizing the  $\Omega$ -background according to

$$\epsilon_1 = \beta^{1/2} \lambda, \quad \epsilon_2 = -\beta^{-1/2} \lambda, \quad (1)$$

we expanded (at small  $q$ )

$$\log Z(a, m, \epsilon_1, \epsilon_2; q) = \sum_{n=-2}^{\infty} \lambda^n \mathcal{G}^{(n)}(a, m, \beta; q) \quad (2)$$

Here,  $a$  are the vectormultiplet moduli,  $m$  parameterizes the masses of flavor hypermultiplets,  $\epsilon_1, \epsilon_2$  are the equivariant parameters of the  $\Omega$ -deformation and  $q$  is the instanton counting parameter.

The expansion (2) is, initially, valid in the weak coupling region ( $a \gg q$ ) of the Seiberg-Witten moduli space  $M$ . Following [3], one can promote the  $\mathcal{G}^{(n)}(a, m, \beta)$  to global, albeit non-holomorphic, objects defined over all of  $M$ . The main result of [1], which generalizes the results of [3], is that the non-holomorphic dependence of the  $\mathcal{G}^{(n)}$  is controlled by the holomorphic anomaly equations familiar from the topological string, for the general value of  $\beta$  (the usual relation to topological string being recovered at  $\beta = 1$ , *i.e.*,  $\epsilon_1 + \epsilon_2 = 0$ ). Moreover, the holomorphic ambiguity can be completely fixed, order by order, by the ( $\beta$ -dependent) singularity structure of the  $\mathcal{G}^{(n)}$ , using local canonical coordinates at each boundary of  $M$ .

A remarkable feature of the results in [1] was the necessity to resort to the extended holomorphic anomaly of [4, 5]. This requirement was apparent in the expansion (2), which generally goes also over odd powers of  $\lambda$ , and hence does not fit into the standard framework in which the  $\mathcal{G}^{(n)}$  are identified with topological string amplitudes  $\mathcal{F}^{(g)}$  and  $n = 2g - 2$  is even. On the other hand, the existence of the odd sector in the expansion (2) appears to be in conflict with the refined BPS expansion

of the topological string partition function proposed in [6]. Indeed, a quick peek reveals that that expansion is manifestly symmetric under  $\lambda \rightarrow -\lambda$ , *i.e.*, expressed in terms of the  $\Omega$ -background [22], we have the symmetry

$$(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2) \quad (3)$$

and hence we should have found  $\mathcal{G}^{(n)} = 0$  whenever  $n$  is odd. The purpose of this note is to release some tension about this point.

In [6], the symmetry (3) was ensured by exploiting the redefinition of flat coordinates (Kähler parameters  $t$  of the A-model) that vanishes in the unrefined limit, schematically  $t \rightarrow t + \delta t$  with  $\delta t \propto (\epsilon_1 + \epsilon_2)$ . As we shall see below, we can in fact restore the symmetry, and remove the extension of the holomorphic anomaly, also in the gauge theory case by shifting the mass parameters

$$m \rightarrow m + (\epsilon_1 + \epsilon_2)/2. \quad (4)$$

This is similar to the shifts in [6], and is in fact related to them via geometric engineering, but differs from them in one crucial respect. Namely, while the variables being shifted in [6] are dynamical fields (moduli), the  $m$  appear as external parameters in the gauge theory. Anticipating some of our conclusions, this means that the information about this *shift of an a priori non-dynamical parameter gets traded in our formalism with the non-trivial extension* of the holomorphic anomaly equation. We will elaborate on this insight below.

The redefinition of mass parameters (4) has played a significant rôle in the recent (“AGT”) relations between four and two-dimensional conformal field theories, see in particular [7, 8]. That the instanton partition function should be invariant under the symmetry (3) was especially emphasized in [9]. We learned in [8] that shifts as in (4) first appeared in [10].

## II. INSTANTON COUNTING

According to [11], [2], the instanton partition function in  $\Omega$ -background for gauge theory with  $N_f$  flavors is given by

$$Z^{\text{inst}}(a, m, \epsilon_1, \epsilon_2; q) = \sum_k q^k \int_{\mathcal{M}_k} \mathbf{e}(V \otimes S) \quad (5)$$

Here,  $\mathcal{M}_k$  is the (compactified) moduli space of instantons of charge  $k$  on  $\mathbb{R}^4$ ,  $V$  is the bundle of solutions of the Dirac equation over it, and  $S \cong \mathbb{C}^{N_f}$  is the flavor space. The integration takes place in the equivariant cohomology  $\mathbb{C}(a, m, \epsilon_1, \epsilon_2)$ . It was further shown in [2] that at a fixed point labelled by a collection of partitions  $Y = (Y_1, Y_2)$  (We will here consider only  $SU(2)$  gauge theory with  $N_f < 4$  fundamental flavors.), the Euler class  $e(V \otimes S)$  localizes to

$$f^Y(a, m, \epsilon_1, \epsilon_2) = \sum_{\gamma=1,2} \prod_{k=1,\dots,N_f} \prod_{s \in Y_\gamma} (a_\gamma + m_k + \epsilon_1(i-1) + \epsilon_2(j-1)), \quad (6)$$

where  $i, j$  are the coordinates of the box  $s \in Y_\gamma$ , and  $a_1 + a_2 = 0$  is implicit. The formula (6) was also found in [12].

By combining the remarks in [9] and the original observations in [10] with the results of [8], those of [7], and the methods of [6], we are led to consider the alternative expression

$$\tilde{f}^Y(a, m, \epsilon_1, \epsilon_2) = \sum_{\gamma=1,2} \prod_{k=1,\dots,N_f} \prod_{s \in Y_\gamma} (a_\gamma + m_k + \epsilon_1(i-1) + \epsilon_2(j-1) + \frac{\epsilon_1 + \epsilon_2}{2}). \quad (7)$$

Several explanations and remarks are in order. First of all, we observe that we may obtain  $\tilde{f}$  from  $f$  by shifting the masses diagonally,

$$m \rightarrow m + (\epsilon_1 + \epsilon_2)/2. \quad (8)$$

Now, as mentioned above, the redefinition (8) is related to the procedure of [6] for ensuring a sensible BPS expansion of the refined topological string partition function. Second, in [7], shifts by  $(\epsilon_1 + \epsilon_2)/2$  as in (8) were used to relate the mass parameters and Coulomb moduli appearing in the instanton partition function with the Liouville momenta labelling a conformal block dual to the instanton partition function [23]. Next, the relation (8) between the mass parameters of [7] (which are those of [2]) and those of the localization computation of [13] for the  $\mathcal{N} = 2^*$  gauge theory on  $S^4$  was found in [8] to ensure the restoration of  $\mathcal{N} = 4$  supersymmetry in the limit  $m \rightarrow 0$ . (Note that this is the mass of an adjoint hypermultiplet.) This was interpreted in [8] as the “physical” definition of the mass parameter.

Finally, in [9], a computation showed that it would be more natural to use the kernel of the Dirac operator instead of the Dolbeault operator coupled to the instanton background in (5) as the definition of the gauge theory with fundamental matters, *i.e.*, to twist the fermions. As usual, this is accomplished by tensoring with the half canonical bundle, a line bundle with weight  $(\epsilon_1 + \epsilon_2)/2$ . In the localization, this is precisely equivalent to the shift (8) of the mass parameters.

For comparison, we find it convenient to interpolate between the two prescriptions (6) and (7) by introducing an additional parameter  $\xi$  continuously tuning the magnitude of the shift. Thus, we consider the family of partition functions

$$Z_\xi(a, m, \epsilon_1, \epsilon_2; q) = Z_1(a, m + (1 - \xi)\frac{\epsilon_1 + \epsilon_2}{2}, \epsilon_1, \epsilon_2; q). \quad (9)$$

Here  $\xi = 1$  corresponds to using (6), as in [1]. The value  $\xi = 0$  corresponds to (7). In general, we find that  $Z_0$  is symmetric (for all values of  $m$ ) under  $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_1, -\epsilon_2)$ , as announced in [9]. We also find that the theory is invariant under  $\xi \rightarrow -\xi$ .

### III. B-MODEL

It was found in [1] that the amplitudes  $\mathcal{G}^{(n)}$  defined as coefficients of  $\lambda^n$  in the expansion (2) of  $Z_1$  (for  $SU(2)$ ,  $N_f = 0, 1, 2, 3$  flavors, with  $m \equiv 0$ ) satisfy, when appropriately continued to modular invariant expressions over the Seiberg-Witten moduli space, the extended holomorphic anomaly equation of [4, 5]. The Griffiths infinitesimal invariant measuring the extension vanishes for  $N_f = 0, 2, 3$ , and for  $N_f = 1$  can be obtained from the chain integral of the Seiberg-Witten differential between an appropriate pair of points on the Seiberg-Witten curve. The singularity structure around monopole/dyon points was enough to completely fix the holomorphic ambiguity.

To give a bit more details, we recall that the full gauge theory partition function  $Z_\xi$  is the product of the instanton part  $Z_\xi^{\text{inst}}$ , discussed above, and a perturbative part  $Z_\xi^{\text{pert}}$ . To write this piece, we introduce as in [1] the two sets of functions  $\Phi^{(n)}(\beta)$  and  $\Psi^{(n)}(\beta)$  by the asymptotic expansion of the two Schwinger integrals

$$\int \frac{ds}{s} \frac{e^{-xs}}{(e^{\epsilon_1 s} - 1)(e^{\epsilon_2 s} - 1)} \sim \dots + \sum_{n \geq 0} \frac{\lambda^n}{x^n} \Phi^{(n)}(\beta) \quad (10)$$

$$\int \frac{ds}{s} \frac{e^{-xs} e^{(\epsilon_1 + \epsilon_2)s/2}}{(e^{\epsilon_1 s} - 1)(e^{\epsilon_2 s} - 1)} \sim \dots + \sum_{n \geq 0} \frac{\lambda^n}{x^n} \Psi^{(n)}(\beta). \quad (11)$$

(The  $\Phi^{(n)}$  are essentially the  $\gamma_{\epsilon_1, \epsilon_2}$  of [2, 14, 15], and the  $\Psi^{(n)}$  appear as  $\delta_{\epsilon_1, \epsilon_2}$  in [9].) We then have for vanishing bare mass of the fundamentals [7, 9]

$$\log Z_0^{\text{pert}} \sim \sum_{n \text{ even}} \lambda^n \left( \frac{2\Phi^{(n)}(\beta)}{(2a)^n} - \frac{2N_f \Psi^{(n)}(\beta)}{a^n} \right). \quad (12)$$

The first term comes from integrating out the 2 vectormultiplets (W-bosons) of BPS mass  $\pm 2a$  in the limit  $a \rightarrow \infty$ , and the second term from the  $2N_f$  hypermultiplets of mass  $\pm a$ .

Similarly, the leading behaviour around a point with massless monopole/dyon is governed by  $\Psi^{(n)}(\beta)$ , corresponding to integrating out a light hypermultiplet with mass given by the local flat coordinate.

We may now repeat the calculations of [1] (which were done for  $\xi = 1$ ), for general value of  $\xi$ . We find that the amplitudes  $\mathcal{G}_\xi^{(n)}(\beta)$  appearing in the expansion of  $\log Z_\xi$  are always governed by the extended holomorphic anomaly equation. In particular, we find that imposing the  $\beta$ - and  $\xi$ -dependent gap structure at the monopole/dyon points completely fixes the leading weak coupling behaviour given by (12) (and its shifts). As observed above, the  $\mathcal{G}^{(n)}$  of  $n$  odd vanish for  $\xi = 0$ , so in this case we use the standard holomorphic anomaly equation, even for  $N_f = 1$ , and 3.

Let us also briefly comment on the case with two flavors. As noted in [1] at  $\xi = 1$ , the  $\mathbb{Z}_2$  symmetry between monopole and dyon point (which obtains in the unrefined case  $\beta = 1$ ), is broken for generic value of  $\xi$ . The explicit calculation shows that this symmetry is in fact restored at  $\xi = 0$ , giving additional corroboration that this is the most symmetric value. Correspondingly, at this value of  $\xi$ , the leading singularities at monopole and dyon points are both captured by the  $\Psi^{(n)}$  coefficients.

Finally, we emphasize again that we have performed these calculations only for vanishing bare mass of the flavor hypermultiplets,  $m = 0$ . It would be interesting to check the massive case as well, and in particular take a look at the various superconformal points in the space of theories.

#### IV. DISCUSSION

In this brief note, we have scouted the freedom of shifting the masses of fundamental hypermultiplets of  $\mathcal{N} = 2$  supersymmetric  $SU(2)$  gauge theory by the self-dual  $\Omega$ -background parameter,  $\epsilon_1 + \epsilon_2$ . We have seen that for all values of the shift parameter  $\xi$ , the deformed partition function  $Z_\xi$  is controlled in the B-model by the extended holomorphic anomaly equation together with appropriate boundary conditions. The value  $\xi = 0$  is preferred by the circumstance that  $Z_0$  is symmetric in  $\lambda \sim \sqrt{\epsilon_1 \epsilon_2}$ , and that the B-model formalism reproduces the perturbative spectrum most precisely. This is also the value of the shift for which the extension vanishes. One may wish to conclude at this point. However, the consistency of the results for  $\xi \neq 0$  (especially,  $\xi = 1$ ) [24], the naturalness of the extension, and general curiosity begs the question: Is there a physical meaning of the shift?

We can obtain some first hints about this question by taking a higher-dimensional perspective, *i.e.*, by embedding the gauge theory in string theory using geometric engineering [16]. For simplicity, let us consider the  $N_f = 1$  theory. Then the relevant geometry is a Hirzebruch surface with attached conifold-like geometry (see for instance [17]). In particular, it has three Kähler parameter  $Q_i = e^{-t_i}$ ,  $i = 1, 2, 3$ . The first of these (the size of the base) controls the geometric engineering decoupling limit, the other (the size of the fiber) becomes identified with the Coulomb modulus  $a$  in this limit, and the last one (the size of the attached conifold), is the mass parameter  $m$  of the gauge theory. Note that the

field theory vev and the mass parameter have a common geometric origin as closed string moduli.

According to the conjectures originating in [2], the gauge theory partition function in general  $\Omega$ -background should correspond in the string theory to an appropriate “refinement” [17] of the topological string amplitudes. According to [6, 17], the spacetime interpretation of this refined topological string should capture BPS state counting taking account of the spin. As of this writing, there is no compelling proposal for the definition of this refined topological string, neither from worldsheet nor from target space field theory in either A- or B-model. In the A-model, however, we have the “refined topological vertex” [6] that allows the computation of refined amplitudes precisely for geometries that engineer  $\mathcal{N} = 2$  gauge theories. In fact, this refined vertex was constructed precisely to match the five-dimensional version of the instanton partition function obtained in [2, 14]. An important aspect of the formalism is the exploitation of the freedom to shift the Kähler parameters before identifying them with physical quantities such as the masses of (refined) BPS states, or field theory vectormultiplet moduli. The fixing of these shifts could provide important hints for completing the refined vertex formalism.

In the engineering setup, of course, the shift of the mass parameter of the gauge theory  $\delta m \propto \epsilon_1 + \epsilon_2$  lifts in the string theory to the shift of the Kähler modulus  $\delta t_3 \propto \epsilon_1 + \epsilon_2$ . This is precisely the type of shift utilized in [6] to ensure that there is a BPS state counting interpretation. It is not hard to check that our preferred value of the shift at  $\xi = 0$  lifts precisely to an even sector only (integer) refined BPS state counting for the geometry engineering  $N_f = 1$  theory. However, we stress that  $Z_\xi$  generally fulfills the extended holomorphic anomaly equation. Only for the specific value  $\xi = 0$  does it reduce to the standard holomorphic anomaly equation. Also, we note that one can obtain the full family  $Z_\xi$  as the effective field theory limit of the refined topological vertex partition functions on the corresponding geometric engineering geometry, with the shift of mass lifting to a shift of the corresponding Kähler modulus.

In the topological string, the shift of  $t_3$  for non-zero values of  $\xi$  leads to a non-trivial odd sector with  $\mathcal{G}^{(-1)} \sim \xi \partial_{t_3} \mathcal{F}^{(0)}$ , with  $\mathcal{F}^{(0)} = \mathcal{G}^{(-2)}$  the standard prepotential. This is of course nothing but a closed string period, and hence leads to a vanishing extension in the holomorphic anomaly equation. This leads to the following speculative interpretation of the shift [25]: In the (to be refined) topological A-model we are shifting the Kähler parameters as

$$t \rightarrow t + N\lambda, \quad (13)$$

with  $N$  some number, and  $\lambda$  the topological string coupling constant. In the (to be refined) mirror B-model this looks as though switching on  $N$  units of flux through the corresponding 3-cycle, *cmp.* [18]. Hence, it might be possible to interpret  $Z_\xi$  as the effective field theory limit of the partition function of the engineering geometry with additional flux switched on, or, in the spirit of flux/brane

duality, with additional D-branes.

Taking the decoupling limit, the closed string Kähler parameter  $t_3$  becomes the non-dynamical mass parameter  $m$ . While it is still true that the shift introduces an odd sector with  $\mathcal{G}^{(-1)} \propto \xi \partial_m \mathcal{F}^{(0)}$ , the latter is no longer a closed period, and hence gives rise, in general, to a non-trivial extension [26]. As a result, we can no longer interpret the shift in terms of a closed string flux. However, we may still interpret it in terms of adding  $N$  background D-branes, very much in the spirit of the original purpose of the extended holomorphic anomaly equation [4]. In fact, keeping track of the “number of boundaries” via the independent “open string” coupling constant  $\propto \xi \lambda$  should allow the reconstruction of the full  $m$ -dependence.

This type of reasoning might become more compelling if we leave aside the gauge theory interpretation, and only focus attention on the structure of the holomorphic anomaly equation and its solutions, following the line of investigation initiated in [19, 20]. Thus, we view the solution of the extended holomorphic anomaly abstractly as an open-closed string wavefunction, with the extension specifying the D-brane background, and two independent parameters  $(\lambda, \xi)$  playing the rôle of closed and open string coupling, respectively. In this language, the main lesson of our discussion is that the open-closed wave-

function  $Z(\lambda, \xi; a, m)$  is equal to a purely closed string wavefunction  $\tilde{Z}(\lambda; a, m + \delta m)$  with a shift  $\delta m \propto \xi$  of the “non-dynamical closed string” field  $m$ .

We emphasize that this type of open-closed string relation is different from the one first proposed in [19], also in the context of the extended holomorphic anomaly equation. As explained in [20], the improved shift of [19] does remove the extension at the level of the holomorphic anomaly equation, but does not reconstruct the known purely closed string (at least not in a recognizable form). See also [21] for further discussion of the shift of [19] in the light of open-closed string correspondence [27].

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  - [22] After completion of [1], we confirmed that the refined BPS invariants of local Calabi-Yau background such as local  $\mathbb{P}^2$  (which is not directly a geometric engineering situation, but related to one via a blow-down) found in [6] can also be computed in our B-model scheme, using the holomorphic anomaly with vanishing extension. This result was also shown in M. x. Huang and A. Klemm, arXiv:1009.112 [hep-th].
  - [23] It is interesting to remark here that AGT found it expedient to divide the gauge theory partition function by a certain  $U(1)$  factor, in addition to restricting to  $a_1 + a_2 = 0$ . This decoupling plays no apparent rôle in our discussion.
  - [24] There is at least one way in which  $\xi = 1$  is also special. Namely, the orientifold of the  $N_f = 2$  theory with  $\xi = 1$ ,  $\beta = 1$  and appropriately chosen masses naturally yields the same theory without orientifold but at  $\beta = 2$  [1].
  - [25] We thank M. Aganagic for pointing us towards this direction.
  - [26] One may derive the pair of points obtained in [1] by studying the deformation of the Seiberg-Witten curve under turning on the masses.
  - [27] The point of view on open-closed relation taken here is closer to the flux shift (13), and was also proposed a while ago by C. Vafa, and S. Shatashvili.